

Lectures on Probability Theory
and Mathematical Statistics
Second Edition

Marco Taboga

Contents

I	Mathematical tools	1
1	Set theory	3
1.1	Sets	3
1.2	Set membership	4
1.3	Set inclusion	4
1.4	Union	5
1.5	Intersection	6
1.6	Complement	6
1.7	De Morgan's Laws	7
1.8	Solved exercises	7
2	Permutations	9
2.1	Permutations without repetition	9
2.1.1	Definition of permutation without repetition	9
2.1.2	Number of permutations without repetition	10
2.2	Permutations with repetition	11
2.2.1	Definition of permutation with repetition	11
2.2.2	Number of permutations with repetition	11
2.3	Solved exercises	12
3	k-permutations	15
3.1	k -permutations without repetition	15
3.1.1	Definition of k -permutation without repetition	15
3.1.2	Number of k -permutations without repetition	16
3.2	k -permutations with repetition	17
3.2.1	Definition of k -permutation with repetition	17
3.2.2	Number of k -permutations with repetition	18
3.3	Solved exercises	19
4	Combinations	21
4.1	Combinations without repetition	21
4.1.1	Definition of combination without repetition	21
4.1.2	Number of combinations without repetition	22
4.2	Combinations with repetition	22
4.2.1	Definition of combination with repetition	23
4.2.2	Number of combinations with repetition	23
4.3	More details	25
4.3.1	Binomial coefficients and binomial expansions	25
4.3.2	Recursive formula for binomial coefficients	25

4.4	Solved exercises	25
5	Partitions into groups	27
5.1	Definition of partition into groups	27
5.2	Number of partitions into groups	28
5.3	More details	29
5.3.1	Multinomial expansions	29
5.4	Solved exercises	30
6	Sequences and limits	31
6.1	Definition of sequence	31
6.2	Countable and uncountable sets	32
6.3	Limit of a sequence	32
6.3.1	The limit of a sequence of real numbers	32
6.3.2	The limit of a sequence in general	34
7	Review of differentiation rules	39
7.1	Derivative of a constant function	39
7.2	Derivative of a power function	39
7.3	Derivative of a logarithmic function	40
7.4	Derivative of an exponential function	40
7.5	Derivative of a linear combination	41
7.6	Derivative of a product of functions	41
7.7	Derivative of a composition of functions	42
7.8	Derivatives of trigonometric functions	43
7.9	Derivative of an inverse function	43
8	Review of integration rules	45
8.1	Indefinite integrals	45
8.1.1	Indefinite integral of a constant function	46
8.1.2	Indefinite integral of a power function	46
8.1.3	Indefinite integral of a logarithmic function	47
8.1.4	Indefinite integral of an exponential function	47
8.1.5	Indefinite integral of a linear combination of functions	47
8.1.6	Indefinite integrals of trigonometric functions	48
8.2	Definite integrals	48
8.2.1	Fundamental theorem of calculus	48
8.2.2	Definite integral of a linear combination of functions	49
8.2.3	Change of variable	50
8.2.4	Integration by parts	51
8.2.5	Exchanging the bounds of integration	51
8.2.6	Subdividing the integral	51
8.2.7	Leibniz integral rule	52
8.3	Solved exercises	52
9	Special functions	55
9.1	Gamma function	55
9.1.1	Definition	55
9.1.2	Recursion	56
9.1.3	Relation to the factorial function	56
9.1.4	Values of the Gamma function	57

9.1.5	Lower incomplete Gamma function	58
9.2	Beta function	59
9.2.1	Definition	59
9.2.2	Integral representations	59
9.3	Solved exercises	61
 II Fundamentals of probability		 67
10	Probability	69
10.1	Sample space, sample points and events	69
10.2	Probability	70
10.3	Properties of probability	71
10.3.1	Probability of the empty set	71
10.3.2	Additivity and sigma-additivity	72
10.3.3	Probability of the complement	72
10.3.4	Probability of a union	73
10.3.5	Monotonicity of probability	73
10.4	Interpretations of probability	74
10.4.1	Classical interpretation of probability	74
10.4.2	Frequentist interpretation of probability	74
10.4.3	Subjectivist interpretation of probability	74
10.5	More rigorous definitions	75
10.5.1	A more rigorous definition of event	75
10.5.2	A more rigorous definition of probability	76
10.6	Solved exercises	76
11	Zero-probability events	79
11.1	Definition and discussion	79
11.2	Almost sure and almost surely	80
11.3	Almost sure events	81
11.4	Solved exercises	82
12	Conditional probability	85
12.1	Introduction	85
12.2	The case of equally likely sample points	85
12.3	A more general approach	87
12.4	Tackling division by zero	90
12.5	More details	90
12.5.1	The law of total probability	90
12.6	Solved exercises	91
13	Bayes' rule	95
13.1	Statement of Bayes' rule	95
13.2	Terminology	96
13.3	Solved exercises	96

14 Independent events	99
14.1 Definition of independent event	99
14.2 Mutually independent events	100
14.3 Zero-probability events and independence	101
14.4 Solved exercises	101
15 Random variables	105
15.1 Definition of random variable	105
15.2 Discrete random variables	106
15.3 Absolutely continuous random variables	107
15.4 Random variables in general	108
15.5 More details	109
15.5.1 Derivative of the distribution function	109
15.5.2 Continuous variables and zero-probability events	109
15.5.3 A more rigorous definition of random variable	109
15.6 Solved exercises	109
16 Random vectors	115
16.1 Definition of random vector	115
16.2 Discrete random vectors	116
16.3 Absolutely continuous random vectors	117
16.4 Random vectors in general	118
16.5 More details	119
16.5.1 Random matrices	119
16.5.2 Marginal distribution of a random vector	119
16.5.3 Marginalization of a joint distribution	120
16.5.4 Marginal distribution of a discrete random vector	120
16.5.5 Marginalization of a discrete distribution	120
16.5.6 Marginal distribution of a continuous random vector	120
16.5.7 Marginalization of a continuous distribution	121
16.5.8 Partial derivative of the distribution function	121
16.5.9 A more rigorous definition of random vector	121
16.6 Solved exercises	121
17 Expected value	127
17.1 Definition of expected value	127
17.2 Discrete random variables	128
17.3 Continuous random variables	129
17.4 The Riemann-Stieltjes integral	130
17.4.1 Intuition	131
17.4.2 Some rules	132
17.5 The Lebesgue integral	133
17.6 More details	134
17.6.1 The transformation theorem	134
17.6.2 Linearity of the expected value	134
17.6.3 Expected value of random vectors	136
17.6.4 Expected value of random matrices	136
17.6.5 Integrability	136
17.6.6 L^p spaces	136
17.6.7 Other properties of the expected value	136

17.7 Solved exercises	136
18 Expected value and the Lebesgue integral	141
18.1 Intuition	141
18.2 Linearity of the Lebesgue integral	143
18.3 A more rigorous definition	144
19 Properties of the expected value	147
19.1 Linearity of the expected value	147
19.1.1 Scalar multiplication of a random variable	147
19.1.2 Sums of random variables	147
19.1.3 Linear combinations of random variables	148
19.1.4 Addition of a constant and a random matrix	148
19.1.5 Multiplication of a constant and a random matrix	149
19.2 Other properties	150
19.2.1 Expectation of a positive random variable	150
19.2.2 Preservation of almost sure inequalities	150
19.3 Solved exercises	151
20 Variance	155
20.1 Definition of variance	155
20.2 Interpretation of variance	155
20.3 Computation of variance	155
20.4 Variance formula	156
20.5 Example	156
20.6 More details	157
20.6.1 Variance and standard deviation	157
20.6.2 Addition to a constant	157
20.6.3 Multiplication by a constant	158
20.6.4 Linear transformations	158
20.6.5 Square integrability	159
20.7 Solved exercises	159
21 Covariance	163
21.1 Definition of covariance	163
21.2 Interpretation of covariance	163
21.3 Covariance formula	164
21.4 Example	164
21.5 More details	166
21.5.1 Covariance of a random variable with itself	166
21.5.2 Symmetry	166
21.5.3 Bilinearity	166
21.5.4 Variance of the sum of two random variables	167
21.5.5 Variance of the sum of n random variables	168
21.6 Solved exercises	169

22 Linear correlation	177
22.1 Definition of linear correlation coefficient	177
22.2 Interpretation	177
22.3 Terminology	178
22.4 Example	178
22.5 More details	180
22.5.1 Correlation of a random variable with itself	180
22.5.2 Symmetry	180
22.6 Solved exercises	181
23 Covariance matrix	189
23.1 Definition	189
23.2 Structure of the covariance matrix	189
23.3 Covariance matrix formula	190
23.4 More details	190
23.4.1 Addition to a constant vector	191
23.4.2 Multiplication by a constant matrix	191
23.4.3 Linear transformations	191
23.4.4 Symmetry	192
23.4.5 Semi-positive definiteness	192
23.4.6 Covariance between linear transformations	192
23.4.7 Cross-covariance	193
23.5 Solved exercises	193
24 Indicator function	197
24.1 Definition	197
24.2 Properties of the indicator function	198
24.2.1 Powers	198
24.2.2 Expected value	198
24.2.3 Variance	198
24.2.4 Intersections	198
24.2.5 Indicators of zero-probability events	199
24.3 Solved exercises	199
25 Conditional probability as a random variable	201
25.1 Partitions of events	202
25.2 Probabilities conditional on a partition	203
25.3 The fundamental property	204
25.4 The fundamental property as a definition	205
25.5 More details	206
25.5.1 Conditioning with respect to sigma-algebras	206
25.5.2 Regular conditional probabilities	207
26 Conditional probability distributions	209
26.1 Conditional probability mass function	210
26.2 Conditional probability density function	213
26.3 Conditional distribution function	215
26.4 More details	216
26.4.1 Conditional distribution of a random vector	216
26.4.2 Joint equals marginal times conditional	216
26.5 Solved exercises	216

27 Conditional expectation	221
27.1 Definition	221
27.2 Discrete random variables	221
27.3 Absolutely continuous random variables	223
27.4 Conditional expectation in general	224
27.5 More details	225
27.5.1 Properties of conditional expectation	225
27.5.2 Law of iterated expectations	225
27.6 Solved exercises	226
28 Independent random variables	229
28.1 Definition	229
28.2 Independence criterion	229
28.3 Independence between discrete variables	231
28.4 Independence between continuous variables	232
28.5 More details	233
28.5.1 Mutually independent random variables	233
28.5.2 Mutual independence via expectations	234
28.5.3 Independence and zero covariance	234
28.5.4 Independent random vectors	235
28.5.5 Mutually independent random vectors	235
28.6 Solved exercises	236
III Additional topics in probability theory	239
29 Probabilistic inequalities	241
29.1 Markov's inequality	241
29.2 Chebyshev's inequality	242
29.3 Jensens's inequality	243
29.4 Solved exercises	244
30 Legitimate probability mass functions	247
30.1 Properties of probability mass functions	247
30.2 Identification of a legitimate pmf	248
30.3 Solved exercises	249
31 Legitimate probability density functions	251
31.1 Properties of probability density functions	251
31.2 Identification of a legitimate pdf	252
31.3 Solved exercises	253
32 Factorization of joint probability mass functions	257
32.1 The factorization	257
32.2 A factorization method	257
33 Factorization of joint probability density functions	261
33.1 The factorization	261
33.2 A factorization method	261

34 Functions of random variables and their distribution	265
34.1 Strictly increasing functions	265
34.1.1 Strictly increasing functions of a discrete variable	267
34.1.2 Strictly increasing functions of a continuous variable	268
34.2 Strictly decreasing functions	269
34.2.1 Strictly decreasing functions of a discrete variable	270
34.2.2 Strictly decreasing functions of a continuous variable	271
34.3 Invertible functions	272
34.3.1 One-to-one functions of a discrete variable	273
34.3.2 One-to-one functions of a continuous variable	273
34.4 Solved exercises	274
35 Functions of random vectors and their distribution	277
35.1 One-to-one functions	277
35.1.1 One-to-one function of a discrete vector	277
35.1.2 One-to-one function of a continuous vector	278
35.2 Independent sums	280
35.3 Known moment generating function	281
35.4 Known characteristic function	281
35.5 Solved exercises	281
36 Moments and cross-moments	285
36.1 Moments	285
36.1.1 Definition of moment	285
36.1.2 Definition of central moment	285
36.2 Cross-moments	285
36.2.1 Definition of cross-moment	285
36.2.2 Definition of central cross-moment	287
37 Moment generating function of a random variable	289
37.1 Definition	289
37.2 Moments and mgfs	290
37.3 Distributions and mgfs	291
37.4 More details	293
37.4.1 Mgf of a linear transformation	293
37.4.2 Mgf of a sum	293
37.5 Solved exercises	294
38 Moment generating function of a random vector	297
38.1 Definition	297
38.2 Cross-moments and joint mgfs	299
38.3 Joint distributions and joint mgfs	300
38.4 More details	301
38.4.1 Joint mgf of a linear transformation	301
38.4.2 Joint mgf of a vector with independent entries	302
38.4.3 Joint mgf of a sum	302
38.5 Solved exercises	303

39 Characteristic function of a random variable	307
39.1 Definition	307
39.2 Moments and cfs	308
39.3 Distributions and cfs	309
39.4 More details	310
39.4.1 Cf of a linear transformation	310
39.4.2 Cf of a sum	310
39.4.3 Computation of the characteristic function	311
39.5 Solved exercises	312
40 Characteristic function of a random vector	315
40.1 Definition	315
40.2 Cross-moments and joint cfs	315
40.3 Joint distributions and joint cfs	317
40.4 More details	317
40.4.1 Joint cf of a linear transformation	317
40.4.2 Joint cf of a random vector with independent entries	318
40.4.3 Joint cf of a sum	318
40.5 Solved exercises	319
41 Sums of independent random variables	323
41.1 Distribution function of a sum	323
41.2 Probability mass function of a sum	325
41.3 Probability density function of a sum	327
41.4 More details	329
41.4.1 Sum of n independent random variables	329
41.5 Solved exercises	329
IV Probability distributions	333
42 Bernoulli distribution	335
42.1 Definition	335
42.2 Expected value	336
42.3 Variance	336
42.4 Moment generating function	336
42.5 Characteristic function	337
42.6 Distribution function	337
42.7 More details	337
42.7.1 Relation to the binomial distribution	337
42.8 Solved exercises	337
43 Binomial distribution	341
43.1 Definition	341
43.2 Relation to the Bernoulli distribution	342
43.3 Expected value	344
43.4 Variance	344
43.5 Moment generating function	345
43.6 Characteristic function	346
43.7 Distribution function	346
43.8 Solved exercises	347

44 Poisson distribution	349
44.1 Definition	350
44.2 Relation to the exponential distribution	350
44.3 Expected value	352
44.4 Variance	353
44.5 Moment generating function	354
44.6 Characteristic function	355
44.7 Distribution function	355
44.8 Solved exercises	356
45 Uniform distribution	359
45.1 Definition	359
45.2 Expected value	359
45.3 Variance	360
45.4 Moment generating function	361
45.5 Characteristic function	361
45.6 Distribution function	362
45.7 Solved exercises	362
46 Exponential distribution	365
46.1 Definition	365
46.2 The rate parameter and its interpretation	366
46.3 Expected value	368
46.4 Variance	368
46.5 Moment generating function	369
46.6 Characteristic function	369
46.7 Distribution function	371
46.8 More details	371
46.8.1 Memoryless property	371
46.8.2 Sums of exponential random variables	372
46.9 Solved exercises	372
47 Normal distribution	375
47.1 The standard normal distribution	376
47.1.1 Definition	376
47.1.2 Expected value	377
47.1.3 Variance	377
47.1.4 Moment generating function	378
47.1.5 Characteristic function	379
47.1.6 Distribution function	380
47.2 The normal distribution in general	381
47.2.1 Definition	381
47.2.2 Relation to the standard normal distribution	382
47.2.3 Expected value	382
47.2.4 Variance	382
47.2.5 Moment generating function	383
47.2.6 Characteristic function	383
47.2.7 Distribution function	384
47.3 More details	384
47.3.1 Multivariate normal distribution	384

47.3.2	Linear combinations of normal random variables	384
47.3.3	Quadratic forms involving normal random variables	385
47.4	Solved exercises	385
48	Chi-square distribution	387
48.1	Definition	387
48.2	Expected value	388
48.3	Variance	388
48.4	Moment generating function	389
48.5	Characteristic function	390
48.6	Distribution function	391
48.7	More details	392
48.7.1	Sums of independent Chi-square random variables	392
48.7.2	Relation to the standard normal distribution	393
48.7.3	Relation to the standard normal distribution (2)	395
48.8	Solved exercises	395
49	Gamma distribution	397
49.1	Definition	397
49.2	Expected value	398
49.3	Variance	398
49.4	Moment generating function	399
49.5	Characteristic function	400
49.6	Distribution function	402
49.7	More details	402
49.7.1	Relation to the Chi-square distribution	402
49.7.2	Multiplication by a constant	403
49.7.3	Relation to the normal distribution	404
49.8	Solved exercises	404
50	Student's t distribution	407
50.1	The standard Student's t distribution	407
50.1.1	Definition	408
50.1.2	Relation to the normal and Gamma distributions	408
50.1.3	Expected value	410
50.1.4	Variance	411
50.1.5	Higher moments	413
50.1.6	Moment generating function	414
50.1.7	Characteristic function	414
50.1.8	Distribution function	414
50.2	The Student's t distribution in general	415
50.2.1	Definition	415
50.2.2	Relation to the standard Student's t distribution	415
50.2.3	Expected value	416
50.2.4	Variance	416
50.2.5	Moment generating function	416
50.2.6	Characteristic function	417
50.2.7	Distribution function	417
50.3	More details	417
50.3.1	Convergence to the normal distribution	417

50.3.2	Non-central t distribution	418
50.4	Solved exercises	418
51	F distribution	421
51.1	Definition	421
51.2	Relation to the Gamma distribution	422
51.3	Relation to the Chi-square distribution	424
51.4	Expected value	425
51.5	Variance	426
51.6	Higher moments	427
51.7	Moment generating function	428
51.8	Characteristic function	428
51.9	Distribution function	428
51.10	Solved exercises	429
52	Multinomial distribution	431
52.1	The special case of one experiment	431
52.1.1	Definition	431
52.1.2	Expected value	432
52.1.3	Covariance matrix	432
52.1.4	Joint moment generating function	433
52.1.5	Joint characteristic function	433
52.2	Multinomial distribution in general	434
52.2.1	Definition	434
52.2.2	Representation as a sum of simpler multinomials	434
52.2.3	Expected value	435
52.2.4	Covariance matrix	435
52.2.5	Joint moment generating function	436
52.2.6	Joint characteristic function	436
52.3	Solved exercises	437
53	Multivariate normal distribution	439
53.1	The standard MV-N distribution	439
53.1.1	Definition	439
53.1.2	Relation to the univariate normal distribution	440
53.1.3	Expected value	441
53.1.4	Covariance matrix	441
53.1.5	Joint moment generating function	442
53.1.6	Joint characteristic function	442
53.2	The MV-N distribution in general	443
53.2.1	Definition	443
53.2.2	Relation to the standard MV-N distribution	444
53.2.3	Expected value	445
53.2.4	Covariance matrix	445
53.2.5	Joint moment generating function	445
53.2.6	Joint characteristic function	446
53.3	More details	446
53.3.1	The univariate normal as a special case	446
53.3.2	Mutual independence and joint normality	446
53.3.3	Linear combinations and transformations	447

53.3.4	Quadratic forms	447
53.3.5	Partitioned vectors	447
53.4	Solved exercises	447
54	Multivariate Student's t distribution	451
54.1	The standard MV Student's t distribution	451
54.1.1	Definition	451
54.1.2	Relation to the univariate Student's t distribution	452
54.1.3	Relation to the Gamma and MV normal distributions	452
54.1.4	Marginals	454
54.1.5	Expected value	455
54.1.6	Covariance matrix	455
54.2	The MV Student's t distribution in general	457
54.2.1	Definition	457
54.2.2	Relation to the standard MV Student's t distribution	457
54.2.3	Expected value	458
54.2.4	Covariance matrix	459
54.3	Solved exercises	459
55	Wishart distribution	461
55.1	Definition	461
55.2	Relation to the MV normal distribution	462
55.3	Expected value	462
55.4	Covariance matrix	463
55.5	Review of some facts in matrix algebra	465
55.5.1	Outer products	465
55.5.2	Symmetric matrices	465
55.5.3	Positive definite matrices	465
55.5.4	Trace of a matrix	466
55.5.5	Vectorization of a matrix	466
55.5.6	Kronecker product	466
V	More about normal distributions	467
56	Linear combinations of normals	469
56.1	Linear transformation of a MV-N vector	469
56.1.1	Sum of two independent variables	470
56.1.2	Sum of more than two independent variables	471
56.1.3	Linear combinations of independent variables	471
56.1.4	Linear transformation of a variable	472
56.1.5	Linear combinations of independent vectors	473
56.2	Solved exercises	473
57	Partitioned multivariate normal vectors	477
57.1	Notation	477
57.2	Normality of the sub-vectors	478
57.3	Independence of the sub-vectors	478

58 Quadratic forms in normal vectors	481
58.1 Review of relevant facts in matrix algebra	481
58.1.1 Orthogonal matrices	481
58.1.2 Symmetric matrices	482
58.1.3 Idempotent matrices	482
58.1.4 Symmetric idempotent matrices	482
58.1.5 Trace of a matrix	483
58.2 Quadratic forms in normal vectors	483
58.3 Independence of quadratic forms	484
58.4 Examples	485
VI Asymptotic theory	489
59 Sequences of random variables	491
59.1 Terminology	491
59.1.1 Realization of a sequence	492
59.1.2 Sequences on a sample space	492
59.1.3 Independent sequences	492
59.1.4 Identically distributed sequences	492
59.1.5 IID sequences	492
59.1.6 Stationary sequences	492
59.1.7 Weakly stationary sequences	493
59.1.8 Mixing sequences	494
59.1.9 Ergodic sequences	494
59.2 Limit of a sequence of random variables	495
60 Sequences of random vectors	497
60.1 Terminology	497
60.1.1 Realization of a sequence	497
60.1.2 Sequences on a sample space	497
60.1.3 Independent sequences	497
60.1.4 Identically distributed sequences	498
60.1.5 IID sequences	498
60.1.6 Stationary sequences	498
60.1.7 Weakly stationary sequences	499
60.1.8 Mixing sequences	499
60.1.9 Ergodic sequences	499
60.2 Limit of a sequence of random vectors	500
61 Pointwise convergence	501
61.1 Sequences of random variables	501
61.2 Sequences of random vectors	502
61.3 Solved exercises	503
62 Almost sure convergence	505
62.1 Sequences of random variables	505
62.2 Sequences of random vectors	507
62.3 Solved exercises	507

63 Convergence in probability	511
63.1 Sequences of random variables	511
63.2 Sequences of random vectors	513
63.3 Solved exercises	514
64 Mean-square convergence	519
64.1 Sequences of random variables	519
64.2 Sequences of random vectors	521
64.3 Solved exercises	522
65 Convergence in distribution	527
65.1 Sequences of random variables	527
65.2 Sequences of random vectors	529
65.3 More details	529
65.3.1 Proper distribution functions	529
65.4 Solved exercises	530
66 Relations between modes of convergence	533
66.1 Almost sure \Rightarrow Probability	533
66.2 Probability \Rightarrow Distribution	533
66.3 Almost sure \Rightarrow Distribution	534
66.4 Mean square \Rightarrow Probability	534
66.5 Mean square \Rightarrow Distribution	534
67 Laws of Large Numbers	535
67.1 Weak Laws of Large Numbers	535
67.1.1 Chebyshev's WLLN	535
67.1.2 Chebyshev's WLLN for correlated sequences	537
67.2 Strong Laws of Large numbers	540
67.2.1 Kolmogorov's SLLN	540
67.2.2 Ergodic theorem	541
67.3 Laws of Large numbers for random vectors	541
67.4 Solved exercises	542
68 Central Limit Theorems	545
68.1 Examples of Central Limit Theorems	546
68.1.1 Lindeberg-Lévy CLT	546
68.1.2 A CLT for correlated sequences	548
68.2 Multivariate generalizations	549
68.3 Solved exercises	550
69 Convergence of transformations	555
69.1 Continuous mapping theorem	555
69.1.1 Convergence in probability of sums and products	555
69.1.2 Almost sure convergence of sums and products	556
69.1.3 Convergence in distribution of sums and products	556
69.2 Slutski's Theorem	557
69.3 More details	557
69.3.1 Convergence of ratios	557
69.3.2 Random matrices	558
69.4 Solved exercises	558

VII	Fundamentals of statistics	561
70	Statistical inference	563
70.1	Samples	563
70.2	Statistical models	565
70.2.1	Parametric models	565
70.3	Statistical inferences	566
70.4	Decision theory	567
71	Point estimation	569
71.1	Estimate and estimator	569
71.2	Estimation error, loss and risk	569
71.3	Other criteria to evaluate estimators	571
71.3.1	Unbiasedness	571
71.3.2	Consistency	571
71.4	Examples	572
72	Point estimation of the mean	573
72.1	Normal IID samples	573
72.1.1	The sample	573
72.1.2	The estimator	573
72.1.3	Expected value of the estimator	573
72.1.4	Variance of the estimator	574
72.1.5	Distribution of the estimator	574
72.1.6	Risk of the estimator	575
72.1.7	Consistency of the estimator	575
72.2	IID samples	575
72.2.1	The sample	575
72.2.2	The estimator	576
72.2.3	Expected value of the estimator	576
72.2.4	Variance of the estimator	576
72.2.5	Distribution of the estimator	576
72.2.6	Risk of the estimator	576
72.2.7	Consistency of the estimator	577
72.2.8	Asymptotic normality	577
72.3	Solved exercises	577
73	Point estimation of the variance	579
73.1	Normal IID samples - Known mean	579
73.1.1	The sample	579
73.1.2	The estimator	579
73.1.3	Expected value of the estimator	580
73.1.4	Variance of the estimator	580
73.1.5	Distribution of the estimator	581
73.1.6	Risk of the estimator	581
73.1.7	Consistency of the estimator	582
73.2	Normal IID samples - Unknown mean	582
73.2.1	The sample	582
73.2.2	The estimator	582
73.2.3	Expected value of the estimator	583
73.2.4	Variance of the estimator	585

73.2.5	Distribution of the estimator	585
73.2.6	Risk of the estimator	587
73.2.7	Consistency of the estimator	588
73.3	Solved exercises	589
74	Set estimation	591
74.1	Confidence set	591
74.2	Coverage probability - confidence coefficient	592
74.3	Size of a confidence set	592
74.4	Other criteria to evaluate set estimators	593
74.5	Examples	593
75	Set estimation of the mean	595
75.1	Normal IID samples - Known variance	595
75.1.1	The sample	595
75.1.2	The interval estimator	595
75.1.3	Coverage probability	596
75.1.4	Confidence coefficient	596
75.1.5	Size	597
75.1.6	Expected size	597
75.2	Normal IID samples - Unknown variance	597
75.2.1	The sample	597
75.2.2	The interval estimator	597
75.2.3	Coverage probability	598
75.2.4	Confidence coefficient	601
75.2.5	Size	601
75.2.6	Expected size	602
75.3	Solved exercises	603
76	Set estimation of the variance	607
76.1	Normal IID samples - Known mean	607
76.1.1	The sample	607
76.1.2	The interval estimator	607
76.1.3	Coverage probability	608
76.1.4	Confidence coefficient	608
76.1.5	Size	609
76.1.6	Expected size	609
76.2	Normal IID samples - Unknown mean	609
76.2.1	The sample	609
76.2.2	The interval estimator	610
76.2.3	Coverage probability	610
76.2.4	Confidence coefficient	611
76.2.5	Size	611
76.2.6	Expected size	611
76.3	Solved exercises	611

77 Hypothesis testing	615
77.1 Null hypothesis	616
77.2 Alternative hypothesis	616
77.3 Types of errors	616
77.4 Critical region	616
77.5 Test statistic	617
77.6 Power function	617
77.7 Size of a test	617
77.8 Criteria to evaluate tests	618
77.9 Examples	618
78 Hypothesis tests about the mean	619
78.1 Normal IID samples - Known variance	619
78.1.1 The sample	619
78.1.2 The null hypothesis	619
78.1.3 The alternative hypothesis	620
78.1.4 The test statistic	620
78.1.5 The critical region	620
78.1.6 The power function	620
78.1.7 The size of the test	621
78.2 Normal IID samples - Unknown variance	621
78.2.1 The sample	621
78.2.2 The null hypothesis	622
78.2.3 The alternative hypothesis	622
78.2.4 The test statistic	622
78.2.5 The critical region	622
78.2.6 The power function	623
78.2.7 The size of the test	625
78.3 Solved exercises	626
79 Hypothesis tests about the variance	629
79.1 Normal IID samples - Known mean	629
79.1.1 The sample	629
79.1.2 The null hypothesis	629
79.1.3 The alternative hypothesis	630
79.1.4 The test statistic	630
79.1.5 The critical region	630
79.1.6 The power function	630
79.1.7 The size of the test	631
79.2 Normal IID samples - Unknown mean	631
79.2.1 The sample	631
79.2.2 The null hypothesis	631
79.2.3 The alternative hypothesis	632
79.2.4 The test statistic	632
79.2.5 The critical region	632
79.2.6 The power function	632
79.2.7 The size of the test	633
79.3 Solved exercises	633